CLASSIFYING ALMOST-DISJOINT FAMILIES WITH APPLICATIONS TO $\beta N - N$

BY
STEPHEN H. HECHLER

ABSTRACT

A family of infinite subsets of the set N of natural numbers will be called almost disjoint iff any two of its members have finite intersection. We shall define such a family $\mathscr{F}$ to be n-separable iff for every decomposition $\mathscr{D} = \{D_1, \cdots D_n\}$ of N into n or fewer disjoint subsets there exist sets $F \in \mathscr{F}$ and $D \in \mathscr{D}$ such that $F \subseteq D$, and we shall use this and related notions to classify almost-disjoint families, using, on occasion, special axioms of set theory.

1. Introduction

For any infinite denumerable set S we define two subsets $F, G \subset S$ to be *almost disjoint* iff their intersection $F \cap G$ is finite, and we define a family $\mathscr{F}$ of subsets of S to be an *almost-disjoint family* iff every member of $\mathscr{F}$ is infinite, $\mathscr{F}$ itself is infinite, and any two distinct members of $\mathscr{F}$ are almost disjoint.

It was first proven by Cantor that no countable maximal almost-disjoint family can be maximal, and it is equally well known that there always exist almost-disjoint families of cardinality $2^{\aleph_0}$. On the other hand the author is not aware of anything being known about the internal structure of such families. This is not surprising, however, when we remember that the proof of the existence of maximal almost-disjoint families appears to require the axiom of choice (which we shall assume, without further mention, throughout this paper), and that even with this axiom, the existence of such maximal families having cardinality less than $2^{\aleph_0}$ is undecidable. This latter is especially important because even though our classification scheme will be well defined for all families of almost-disjoint sets (and, in fact, all families of subsets of some base set), many of our stronger existence theorems will require the hypothesis that all maximal almost-disjoint families have

Received April 21, 1971 and in revised form July 16, 1971.

cardinality $2^{\aleph_0}$. Elsewhere [4] we shall prove that it is consistent that this hypothesis be false; here we shall prove (9.3) that it is a consequence of Martin's axiom [5] and therefore the continuum hypothesis. This will allow us to conclude that at least in certain models of set theory, large classes of almost-disjoint families do exist. We shall conclude, in Section 10, with some applications to $\beta N - N$.

At this point we wish to acknowledge our debt to Gustave Choquet whose two papers [1,2] were the inspiration for this work.

2. Separability

We begin with some additional notation. If f is any function and A and B are any sets, then dm$f$ will be used to denote the domain of f, cd$A$ will denote the cardinality of A, ${}^{A}B$ will denote the set of all functions from A into B, B^{A} will denote cd$({}^{A}B)$, and $f[A]$ will denote the set $\{f(a): a \in A\}$. We shall refer to f as being *complete* if for each b in the range of f we have $\mathrm{cd}\{a: f(a) = b\} = \mathrm{cd}(\mathrm{dm}f)$.

Also, for any set S, $\boldsymbol{P}(S)$ will denote the power set of the power set of S, $\boldsymbol{A}(S)$ will denote the set of all almost-disjoint families of subsets of S, and $\boldsymbol{M}(S)$ will denote the set of all maximal members of $\boldsymbol{A}(S)$. For any two families $\mathscr{F}, \mathscr{G} \in \boldsymbol{P}(S)$, let $\mathscr{F} / \mathscr{G}$ denote the family $\{F \in \mathscr{F}: \exists G \in \mathscr{G}\ F \subseteq G\}$. When no confusion can result, we shall usually write $\mathscr{F}/M$ rather than $\mathscr{F}/\{M\}$. A family $\mathscr{D} \in \boldsymbol{P}(S)$ will be called a *decomposition* of S iff it is finite, its members are pairwise disjoint and its union is S. For any set S and each natural number n we shall use $\boldsymbol{D}_n(S)$ to denote the set of all n or fewer element decompositions of S and we shall use $\boldsymbol{D}(S)$ to denote the set of all such decompositions. Since, with only one exception (9.6), the properties of the families which we construct will not depend upon the specific set S with which we begin, we shall generally use the set of natural numbers for this purpose and shall reserve the symbol N for this set. Whenever we speak of an almost-disjoint family without specifically stating over which set it is taken, we shall be referring to one taken over N, i.e. a member of $\boldsymbol{A}(N)$.

Our main tool for classification will be the notion of "separability". We define a family $\mathscr{F} \in \boldsymbol{P}(S)$ to *separate* over a decomposition $\mathscr{D} \in \boldsymbol{D}(S)$ iff $\mathscr{F}/\mathscr{D}$ is not empty, i.e., iff at least one member of $\mathscr{F}$ is completely contained in some member of the decomposition. In applications there will often be not only one such member, but infinitely many. When we wish to distinguish, we shall refer to $\mathscr{F}$ as *weakly* or *strongly* separating over $\mathscr{D}$ depending upon whether $\mathscr{F}/\mathscr{D}$ is finite or infinite. Finally, we extend these notions by defining a family $\mathscr{F} \in \boldsymbol{P}(S)$ to be (*strongly*) *n-separable* iff it (strongly) separates over every decomposition $D \in \boldsymbol{D}_n(S)$.

These notions can be generalized to decompositions of n-tuples in a natural way. For any set A let $[A]^m$ be the set of all m element subsets of A, and for any family $\mathscr{F} \in \boldsymbol{P}(S)$ let $|\mathscr{F}|^n = \{[F]^n : F \in \mathscr{F}\}$. Then Ramsey's theorem [6] can be thought of as saying that if $\mathscr{F}$ is the family of all infinite subsets of N and $\mathscr{D}$ is any decomposition of $[N]^n$ (i.e. $\mathscr{D} \in \boldsymbol{D}([N]^n)$), then $|\mathscr{F}|^n / \mathscr{D} \neq \emptyset$. It therefore seems reasonable to refer to a family $\mathscr{F}$ as (strongly) *m-n-separable* iff for each decomposition $\mathscr{D} \in \boldsymbol{D}_n([N]^m)$ the family $|\mathscr{F}|^n / \mathscr{F}$ is nonempty (infinite), and *fully Ramsey* iff it is strongly m-n-separable for every m and n belonging to N. Our notions of n separability then reduce to 1-n-separability. However, in this paper we shall not treat the general case but shall confine ourselves to the aforementioned special case with the exception of an occasional mention of fully Ramsey families.

We begin our task of classifying families by proving that there is at most one number n for which a family is n-separable but not strongly n-separable.

THEOREM. 2.1. *Every $n+1$-separable family $\mathscr{F} \in \boldsymbol{P}(N)$ containing only infinite members is strongly n-separable.*

PROOF. Assume there exists a decomposition $\mathscr{D} \in \boldsymbol{D}_n(N)$ such that $\mathscr{F} / \mathscr{D}$ is finite. Then there exists a finite set E containing at least one point from each member of $\mathscr{F} / \mathscr{D}$. But now, $\mathscr{F}$ does not separate over

$$(\{D - E : D \in \mathscr{D}\} \cup \{E\}) \in \boldsymbol{D}_{n+1}(N). \square$$

Thus any family $\mathscr{F} \in \boldsymbol{P}(N)$ falls into one of the following categories.

2.2.1. For every $n \in N$, $\mathscr{F}$ is strongly n-separable.

2.2.2. There exists an $n \in N$ such that $\mathscr{F}$ is strongly n-separable but is not $n+1$-separable.

2.2.3. There exists an $n \in N$ such that $\mathscr{F}$ is n-separable but not strongly n-separable, and therefore by 2.1, is not $n+1$-separable but is strongly m-separable for every $m < n$.

We shall refer to families satisfying 2.2.1 as *fully separable*, those satisfying 2.2.2 as *sharply n-separable*, and those satisfying 2.2.3 as *weakly n-separable*.

Although all families in $\boldsymbol{P}(N)$ can be classified using these notions, we shall be interested only in almost-disjoint families and, in particular, maximal almost-disjoint families. Unfortunately, maximality will not be of any great use because:

THEOREM 2.3. *Every weakly n-separable, sharply n-separable, fully separable, or fully Ramsey family $\mathscr{F} \in \boldsymbol{A}(N)$ can be extended to a weakly n-separable,* *sharply n-separable, fully separable, or fully Ramsey family* $\mathscr{G} \in M(N)$, *respectively.*

PROOF. If $\mathscr{F}$ is fully separable or fully Ramsey, then any extension $\mathscr{G} \in M(N)$ will do. As the remaining two cases are similar to each other, we shall prove only one. Suppose, for example, that $\mathscr{F} \in A(N)$ is weakly n-separable. Let $\mathscr{H}$ be any family such that $\mathscr{F} \cup \mathscr{H} \in M(N)$, let $\mathscr{D} \in D_n(N)$ be any n element decomposition such that $\mathscr{F}/\mathscr{D}$ is finite, and let $E \subset N$ be any finite set such that E intersects every member of $\mathscr{D}$. Then $\mathscr{G} = \mathscr{F} \cup \{H \cup E: H \in \mathscr{H}\}$ is also weakly n-separable and $\mathscr{F} \subseteq \mathscr{G} \in M(N)$.□

3. Absolute existence

We do not know if the existence of all the types of families mentioned in 2.2. follows from the standard axioms of set theory ($ZF + AC$) nor do we know whether the existence of one type of family necessarily implies the existence of any or all other types. In this section we shall present two partial results; later, in Section 8, we shall show that much more is possible if we allow additional axioms. We begin by exhibiting, for reference, an almost-disjoint family of cardinality $2^{\aleph_0}$. The existence of such a family is well known.

CONSTRUCTION 3.1. For each real number r let $[r]$ be the greatest integer less than r, and let $A_r = \{[10^n r]: n \in N\}$. Then $\mathscr{F} = \{A_r: 1 \leqq r < 10\}$ is an almost-disjoint family of cardinality $2^{\aleph_0}$. □

Using this we now have:

THEOREM 3.2. *There exists a fully Ramsey almost-disjoint family* $\mathscr{F}$ *such that for each* $m \in N$ *and each decomposition* $\mathscr{D} \in D([N]^m)$ *the family* $|\mathscr{F}|^m/\mathscr{D}$ *has cardinality* $2^{\aleph_0}$.

PROOF. Let $\mathscr{G} = \{G_\alpha: \alpha \in 2^{\aleph_0}\}$ be any almost-disjoint family of cardinality $2^{\aleph_0}$ (in particular $\mathscr{G}$ may be the family constructed in 3.1), and let f be any complete function from $2^{\aleph_0}$ onto $\cup\{D([N]^m): m \in N\}$. For each $\alpha \in 2^{\aleph_0}$ we may choose, using Ramsey's theorem, an infinite set $F_\alpha \subseteq G_\alpha$ such that if $f(\alpha) \in D([N]^m)$, then for some $D \in f(\alpha)$ we have $[F_\alpha]^m \subseteq D$. Then the family $\mathscr{F} = \{F_\alpha: \alpha \in 2^{\aleph_0}\}$ has the desired properties. □

We also have:

THEOREM 3.3. *If there exists a family* $\mathscr{F} \in A(N)$ *which is* $n+1$*-separable but not* $n+2$*-separable, then there exists a sharply* n*-separable family* $\mathscr{G} \in M(N)$.

PROOF. First assume that $\mathscr{F}$ is only weakly $n+1$-separable. Then there exists a decomposition $\mathscr{D} \in \boldsymbol{D}_{n+1}(N)$ such that $\mathscr{F}/\mathscr{D}$ is finite. But the family $\mathscr{H} = \mathscr{F} - \mathscr{F}/\mathscr{D}$ differs from $\mathscr{F}$ by only finitely many members, so it must strongly separate over exactly the same decompositions as $\mathscr{F}$ does. Thus, because it doesn't separate over $\mathscr{D}$, it must be sharply n-separable. Theorem 2.3 now allows us to extend $\mathscr{H}$ to a sharply n-separable family $\mathscr{G} \in \boldsymbol{M}(N)$.

Now suppose $\mathscr{F}$ is sharply $n+1$-separable and choose a decomposition $\mathscr{D} \in \boldsymbol{D}_{n+2}(N)$ over which $\mathscr{F}$ does not separate and a set $D \in \mathscr{D}$. $\mathscr{F}$ must be at least strongly 2-separable so the family $\mathscr{H} = \mathscr{F}/(N-D)$ must be infinite. Choose any $\mathscr{E} \in \boldsymbol{D}_n(N-D)$. $\mathscr{E} \cup \{D\} \in \boldsymbol{D}_{n+1}(N)$ so $\mathscr{F}/(\mathscr{E} \cup \{D\})$ is infinite. But $\mathscr{F}/D = \varnothing$ so $\mathscr{F}/\mathscr{E} \cup \{D\} = \mathscr{F}/\mathscr{E} = \mathscr{H}/\mathscr{E}$ must be infinite. Thus $\mathscr{H}$ is a strongly n-separable member of $\boldsymbol{A}(N-D)$. However, $\mathscr{H}/(\mathscr{D} - \{D\}) \subseteq \mathscr{F}/\mathscr{D} = \varnothing$ so $\mathscr{H}$ is not $n+1$-separable and is therefore sharply n-separable (over $N-D$). Now let χ be any bijection from $N-D$ onto N and let $\mathscr{H}^* = \{\chi[H] : H \in \mathscr{H}\}$. $\mathscr{H}^*$ is clearly a sharply n-separable member of $\boldsymbol{A}(N)$ and can therefore, by 2.3, be extended to a maximal such family. □

4. Disjoint sets

Since almost-disjointness is simply a generalization of disjointness, we should ask whether almost-disjoint families contain non-singleton disjoint subfamilies, i.e., non-singleton subfamilies whose members are pairwise disjoint. In general the answer is no; take any family and add one fixed element of the underlying set to each of its members. However, the situation with respect to families having separability properties is quite different.

THEOREM 4.1. *Let $\mathscr{F} \in \boldsymbol{A}(N)$ be strongly 2-separable and let $\mathscr{G}$ be any finite disjoint subfamily of $\mathscr{F}$. Then there exists an infinite disjoint family $\mathscr{H}$ such that $\mathscr{G} \subset \mathscr{H} \subset \mathscr{F}$.*

PROOF. It is sufficient to show that every finite disjoint subfamily can be extended to larger disjoint subfamily. Thus let $\mathscr{G} \subseteq \mathscr{F}$ be a finite disjoint family, and let $\mathscr{D} = \{\cup\mathscr{G}, N - \cup\mathscr{G}\} \in \boldsymbol{D}_2(N)$. Because $\mathscr{F}$ is strongly 2-separable $\mathscr{F}/\mathscr{D} = \mathscr{F}/\cup\mathscr{G} \cup \mathscr{F}/(N - \cup\mathscr{G})$ is infinite. However, because $\mathscr{G}$ is finite, $\mathscr{F}/\cup\mathscr{G}$ must be $\mathscr{G}$ itself, and therefore $\mathscr{F}/(N - \cup\mathscr{G})$ must be infinite. But, for any $H \in \mathscr{F}/(N - \cup\mathscr{G})$, the family $\mathscr{G} \cup \{H\}$ is a disjoint subfamily of $\mathscr{F}$ which extends $\mathscr{G}$. □

COROLLARY 4.2. *If $\mathscr{F} \in A(N)$ is strongly 2-separable, then for each $F \in \mathscr{F}$:*

a) *there exists a set $G \in \mathscr{F}$ such that $F \cap G = \varnothing$,*

b) *there exist infinitely many $G \in \mathscr{F}$ such that $F \cap G = \varnothing$,*

c) *there exists an infinite disjoint $\mathscr{G}$ such that $F \in \mathscr{G} \subset \mathscr{F}$.*

Later (8.4), we shall prove, using the additional hypothesis that every member of $M(N)$ has cardinality $2^{\aleph_0}$, that neither 4.1 nor even 4.2a can be extended to weakly 2-separable families. We do not know whether every weakly 2-separable family contains a disjoint pair of elements; we conjecture that it does. We do have some positive results for weakly 2-separable families however; these are somewhat similar to 4.2 but modulo finite sets.

THEOREM 4.3. *If $\mathscr{F} \in A(N)$ is 2-separable, then:*

a) *for each $F \in \mathscr{F}$ there exist infinitely many sets $G \in \mathscr{F}$ such that $F \cap G$ contains at most one point,*

b) *there exists an infinite subfamily $\mathscr{G} \subset \mathscr{F}$ and a finite set $M \subset N$ such that for any two distinct members $F, G \in \mathscr{G}$ we have $F \cap G = M$.*

PROOF. We first construct, by induction, a sequence $\{H_i : i \in N\}$ of distinct members of $\mathscr{F}$ and a sequence of finite sets E_i as follows. Choose any member F of $\mathscr{F}$ as H_1 and let $E_1 = \{k\}$ where k is any member of H_1. Now assume we have already chosen $H_1, \cdots, H_n \in \mathscr{F}$ and finite sets $E_1 \subseteq E_2 \subseteq \cdots \subseteq E_n$ such that for each $i < j \leqq m \leqq n$ we have $H_i \cap H_j \subseteq E_i$ and $\varnothing \neq H_j \cap E_m \subseteq E_j$. Let $\mathscr{D}$ be the decomposition $\{\cup_{i \leqq n} H_i - E_n, E_n \cup (N - \cup_{i \leqq n} H_i)\}$. Let H_{n+1} be any member of $\mathscr{F} / \mathscr{D}$ (which, by the 2-separability of $\mathscr{F}$, is not empty). We see immediately that H_{n+1} must be a subset of $E_n \cup (N - \cup_{i \leqq n} H_i)$. If $H_{n+1} \cap E_n$ is not empty, let $E_{n+1} = E_n$; otherwise, let $E_{n+1} = E_n \cup \{m\}$ where m is any member of H_{n+1}. Now let $\mathscr{H} = \{H_n : n \in N\}$ and let $E = \cup \{E_n : n \in N\}$. We first note that for each $n > 1$ we have $H_n \cap H_1 \subseteq E_1 = \{k\}$; therefore we have satisfied part a of our theorem. For part b. we distinguish two cases.

Case 1. E is finite. Then for some $M \subseteq E$ the family $\mathscr{G} = \{H \in \mathscr{H} : H \cap E = M\}$ is infinite. From the construction it easily follows that for any distinct F, $G \in \mathscr{G}$ we have $F \cap G = F \cap G \cap E = (F \cap E) \cap (G \cap E) = M \cap M = M$.

Case 2. E is infinite. It follows directly from the construction that if $E_{n+1} \neq E_n$, then for every $i \leqq n$ we have $H_{n+1} \cap H_i \subseteq H_{n+1} \cap E_n = \varnothing$. Hence the set $\mathscr{G} = \{H_{i+1} : E_{i+1} \neq E_i\}$ is pairwise disjoint. Thus we may set $M = \varnothing$. □

5. Complete separability

Although full separability appears to be a very strong property, we shall introduce an even stronger one; one which in a sense is maximal. Suppose $\mathscr{F} \in A(N)$, $M \subseteq N$, and for some *finite* family $\mathscr{G} \subseteq \mathscr{F}$ we have $M \subset \cup \mathscr{G}$. Then we shall say that $\mathscr{F}$ *finitely dominates* M. Clearly, in this case, $\mathscr{F}/M$ may be empty and, in fact, one can always find sets M for which it will be. We shall define a family $\mathscr{F} \in A(N)$ to be *completely separable* iff this is the only case in which $\mathscr{F}/M$ is empty, i.e. iff for every $M \subseteq N$ either $\mathscr{F}$ finitely dominates M or $\mathscr{F}/M$ is not empty. Equivalently, we are defining a family $\mathscr{F} \in A(N)$ to be completely separable iff every set $M \subseteq N$ either contains a member of $\mathscr{F}$ as a subset or is itself a subset of the union of a finite subfamily of $\mathscr{F}$. We first note that complete separability is at least as strong as full separability. We do not know if complete separability implies full Ramsey.

THEOREM. 5.1. *If $\mathscr{F} \in A(N)$ is completely separable, then it is fully separable.*

PROOF. Let $\mathscr{D} \in \boldsymbol{D}(N)$. $\mathscr{F}$ cannot finitely dominate every $D \in \mathscr{D}$ and still be an infinite almost-disjoint family so by complete separability it must separate over $\mathscr{D}$. Thus $\mathscr{F}$ is n-separable for every $n \in N$. □

On the other hand, if we remove one member from a fully separable family or one element from every member of such a family, the resulting family will remain fully separable. Thus we see from the following theorem that full separability does not imply complete separability.

THEOREM 5.2. *If $\mathscr{F} \in A(N)$ is completely separable, then $\mathscr{F} \in \boldsymbol{M}(N)$ and $\cup\mathscr{F} = N$.*

PROOF. Suppose $\mathscr{F} \in A(N)$ is completely separable and M is any infinite subset of N which is not a member of $\mathscr{F}$. If $\mathscr{F}$ finitely dominates M, then clearly $\mathscr{F} \cup \{M\} \notin A(N)$; on the other hand if $\mathscr{F}$ doesn't finitely dominate M, then by complete separability there exists an $F \in \mathscr{F}$ such that $F \subset M$, so again $\mathscr{F} \cup \{M\}$ is not almost disjoint. Thus $\mathscr{F}$ is maximal. Similarly, we note that if there exists an $n \in N - \cup \mathscr{F}$, then $\mathscr{F}$ cannot finitely dominate $\{n\}$ nor can any member of $\mathscr{F}$ be a subset of $\{n\}$. □

We stated earlier that complete separability was, in a sense at least, the strongest possible type of separability. However, it would appear that we could strengthen the property by requiring that $\mathscr{F}/M$ be infinite for every set M not finitely dominated by $\mathscr{F}$. We now show that this actually follows from our present definition.

THEOREM 5.3. *If $\mathscr{F} \in \boldsymbol{M}(N)$ is completely separable and M is any subset of N, then $\mathscr{F}/M$ is infinite iff $\mathscr{F}$ doesn't finitely dominate M.*

PROOF. Suppose $\mathscr{F}/M$ is finite. Since $\mathscr{F}$ is completely separable and no member of $\mathscr{F}$ can be a subset of $M - \cup \mathscr{F}/M$, $\mathscr{F}$ must finitely dominate $M - \cup \mathscr{F}/M$. Thus for some finite subfamily $\mathscr{G} \subset \mathscr{F}$ we have $M - \cup \mathscr{F}/M \subseteq \cup \mathscr{G}$ which in turn implies that $M \subseteq \cup(\mathscr{F}/M \cup \mathscr{G})$. Hence $\mathscr{F}$ finitely dominates M. □

6. Star separability

Since the families which we consider are only *almost* disjoint, it seems reasonable to consider also, notions such as "almost separable". For example let $\mathscr{F} \in \boldsymbol{M}(N)$ be completely separable, and let $\mathscr{G} = \{F \cup \{1\} : F \in \mathscr{F}\}$. While it is clear that $\mathscr{G}$ is not even 2-separable, it is also clear that it should be considered as having *some* "separability" property. Thus while $\mathscr{G}$ may not separate over some decomposition $\mathscr{D} \in \boldsymbol{D}(N)$, there will nevertheless exist sets $G \in \mathscr{G}$ and $D \in \mathscr{D}$ such that $\mathscr{D}$ is "almost" a subset of D. It is this property of $\mathscr{G}$ and $\mathscr{D}$ which we wish to explore.

We begin by introducing some new notation. For any two sets A and B we set $A \subseteq {}^*B$ iff $A - B$ is finite and $A = {}^*B$ iff $A \subseteq {}^*B$ and $B \subseteq {}^*A$. Using these, we define, for any two families $\mathscr{F}, \mathscr{G} \in \boldsymbol{P}(N)$, the family

$$\mathscr{F}/{}^*\mathscr{G} = \{F \in \mathscr{F} : \exists G \in \mathscr{G}\, F \subseteq {}^*G\},$$

and if $\mathscr{F}/{}^*\mathscr{G} \neq \emptyset$, we say that $\mathscr{F}$ *separates over $\mathscr{G}$. Finally, we define a family $\mathscr{F} \in \boldsymbol{P}(N)$ to be n-*separable iff it *separates over every decomposition $\mathscr{D} \in \boldsymbol{D}_n(N)$.

We note that this definition is somewhat arbitrary. We could, just as easily, have generalized the definition of a decomposition to have allowed its members to have nonempty albeit finite intersections, and we could have required only that $\mathscr{F}/{}^*\mathscr{D} \neq \emptyset$ for all but finitely many members of $\boldsymbol{D}_n(N)$. However, it is easily seen that changes of this kind in the definitions would in no way affect the n-*separability of any family. An apparently more significant change would be to require in the definition of *separability that $\mathscr{F}/{}^*\mathscr{G} \neq {}^*\emptyset$. However, even this would not affect the n-*separability of almost-disjoint families as can be seen by:

THEOREM. 6.1. *If $\mathscr{F} \in \boldsymbol{A}(N)$ is n-*separable, then for every $\mathscr{D} \in \boldsymbol{D}_n(N)$ the family $\mathscr{F}/{}^*\mathscr{D}$ is infinite.*

PROOF. Suppose $\mathscr{F}$ is n-*separable. Let $M = \{k \in N$: For some $\mathscr{D} \in \boldsymbol{D}_n(N)$ the set $\mathscr{F} /{*}\mathscr{D}$ has exactly k elements$\}$. If M is not empty, it must contain a least element m. Choose some decomposition $\mathscr{D} = \{D_1, \cdots, D_n\}$ such that $\mathscr{F} /{*}\mathscr{D} = \{F_1, \cdots, F_m\}$ has exactly m elements. We may assume without loss of generality that $F_1 \subseteq {*}D_1$. Now choose an infinite set $G \subset D_1 \cap F_1$ such that $G \neq{*} F_1$ and let $\mathscr{E}$ be the decomposition $\{D_1 - G,\ D_2 \cup G,\ D_3, \cdots, D_n\}$. We note that $\mathscr{F} /{*}(D_2 \cup G) = \mathscr{F} /{*}D_2$. Suppose not. Then there must exist a set $F \in \mathscr{F}$ such that $F \nsubseteq{*} D_2$ but $F \subseteq{*} D_2 \cup G$. But this implies that $F \cap G \neq{*} \varnothing$. Hence $F \cap F_1 \neq{*} \varnothing$ so by the almost disjointness of $\mathscr{F}$, F must be F_1. But $F_1 - G \neq{*} \varnothing$ and $F_1 \cap D_2 = {*}\varnothing$ so $F_1 \nsubseteq{*} D_2 \cup G$.

Thus it follows that $\mathscr{F} /{*}\mathscr{E} \subseteq \mathscr{F} /{*}\mathscr{D}$. On the other hand, F_1 cannot belong to $\mathscr{F} /{*}\mathscr{E}$ so $\mathscr{F} /{*}\mathscr{E}$ while not empty because $\mathscr{F}$ is n-*separable, nevertheless has fewer than m elements. But m was chosen to be the smallest member of M so our assumption that M was not empty was false and the theorem is proven. □

Defining full and complete *separability in the obvious manner we see immediately that any fully or completely separable family is also fully or completely *separable respectively. On the other hand, the preceding theorem shows us that there is no *separability notion analogous to weak n-separability. Sharp n-*separability (a family $\mathscr{F}$ will be called *sharply n–*separable* iff it is n-*separable but not $n+1$-*separable) however is more interesting. Since we are especially interested in maximal families, we consider the natural analog to 2.3. The proof turns out to be rather more difficult.

THEOREM 6.2. *If $\mathscr{F} \in A(N)$ is sharply n-*separable, then there exists a sharply n-*separable family $\mathscr{G}$ such that $\mathscr{F} \subseteq \mathscr{G} \in \boldsymbol{M}(N)$.*

PROOF. Let $\mathscr{D} = \{D_0, D_1, \cdots, D_n\}$ be any $n+1$ element decomposition such that $\mathscr{F} /{*}\mathscr{D} = \varnothing$, and choose any family $\mathscr{H}$ such that $\mathscr{F} \subseteq \mathscr{H} \in \boldsymbol{M}(N)$. $\mathscr{H}$ will, of course, remain n-*separable, but we cannot assume that it will necessarily remain *sharply* n-*separable. We therefore proceed to alter $\mathscr{H}$ and possibly $\mathscr{D}$ in such a way as to insure that the resulting family does not *separate over the resulting decomposition. If $H \in \mathscr{H}$ and $\mathscr{A} \in \boldsymbol{M}(H)$, then $(\mathscr{H} - \{H\}) \cup \mathscr{A}$ will still belong to $\boldsymbol{M}(N)$, so we may assume without loss of generality that for each $D \in \mathscr{D}$ either $\mathscr{H} /{*}D$ is empty or it has cardinality $2^{\aleph_0}$. Let $\mathscr{C} = \{D \in \mathscr{D}\colon \mathscr{H} /{*}D \neq \varnothing\}$. We distinguish three cases. If $\mathscr{C} = \varnothing$, then we simply let $\mathscr{G} = \mathscr{H}$. If $\mathscr{C}$ has at least two elements, then we may assume without loss of generality that it is equal to $\{D_0, \cdots, D_m\}$. Thus we can express $\mathscr{H} /{*}\mathscr{D}$ as $\{H^i_\alpha\colon i \leqq m,\ \alpha \in 2^{\aleph_0}\}$ where $H^i_\alpha \subseteq D_i$.

In this case let $\mathscr{G} = (\mathscr{H} - \mathscr{H}/^*\mathscr{D}) \cup \{\cup_{i\leqq m} H^i_\alpha : \alpha \in 2^{\aleph_0}\}$. Finally, suppose $\mathscr{C}$ has exactly one element, which we may assume is D_0. Choose any set $H \in \mathscr{H}/^*D_0$, any infinite set $I \subset H \cap D_0$ such that $I \neq^* H$, and any family $\mathscr{I} = \{I_\alpha : \alpha \in 2^{\aleph_0}\} \in \boldsymbol{M}(I)$. Because $\mathscr{F}/^*D_0 \subseteq \mathscr{F}/^*\mathscr{D} = \emptyset$, we have $\mathscr{F} \subseteq \mathscr{H} - \{H\}$. We also note that $\mathscr{H}/^*(D_1 \cup I)$ must be empty. Suppose otherwise and let $F \in \mathscr{H}/^*(D_1 \cup I)$. But $\mathscr{H}/^*D_1 = \emptyset$, so $F \cap I \subseteq F \cap H$ must be infinite. Thus $F = H$. However, I was chosen such that $H \cap (D_0 - I)$ remained infinite, so H cannot belong to $\mathscr{H}/^*(D_1 \cup I)$. We also note that the family $\mathscr{H}/^*(D_0 - I) \cup \{H - I\}$ is equal to $(\mathscr{H}/^*D_0 - \{H\}) \cup \{H - I\}$ and can therefore be written in the form $\{J_\alpha : \alpha \in 2^{\aleph_0}\}$. Now let $\mathscr{G} = (\mathscr{H} - \mathscr{H}/^*D_0) \cup \{I_\alpha \cup J_\alpha : \alpha \in 2^{\aleph_0}\}$, and let $\mathscr{E} = \{D_0 - I, D_1 \cup I, D_2, \cdots, D_n\}$. Since $\mathscr{F}/^*D_0 = \emptyset$ we have $\mathscr{F} \subseteq \mathscr{G}$, and thus $\mathscr{G}$ is n-*separable. On the other hand $\mathscr{G}/^*\mathscr{E} = \emptyset$ so $\mathscr{G}$ is not $n+1$-*separable. □

Later (Theorem 8.3) we shall show that, under our usual hypothesis, there exist sharply n-*separable families for all $n \in N$. Here we content ourselves with proving that the existence of a sharply n-*separable family implies the existence of sharply m-*separable families for all $m \leqq n$.

THEOREM 6.3. *If there exists a sharply $n+1$-*separable family $\mathscr{F} \in A(N)$, then there exists a sharply n-*separable family $\mathscr{G} \in \boldsymbol{M}(N)$.*

PROOF. Let $\mathscr{D} \in \boldsymbol{D}_{n+2}(N)$ be any decomposition over which $\mathscr{F}$ doesn't *separate and let D be any member of $\mathscr{D}$. Then, as in the proof of 3.3 we look at $\mathscr{H} = \{F \in \mathscr{F} : F \subseteq^* N - D\}$. $\mathscr{F}$ must be at least 2-*separable, so $\mathscr{H}$ must be infinite. It follows easily that $\mathscr{H} \in A(N - D)$ and is sharply n-*separable with respect to $N - D$. As earlier, we first convert $\mathscr{H}$ to a similar family over N and then, if necessary, use 6.2 to extend it to the desired maximal family. □

7. Connectedness

In another direction, we can ask about connectedness properties; i.e. given any two (or three or four, etc.,) points, is there a member of the family which contains them both? We define a family $\mathscr{F} \in \boldsymbol{P}(N)$ to be *(strongly) n-connected* iff for every set $S \subset N$ containing n elements the family $\{F \in \mathscr{F} : S \subset F\}$ is (infinite) nonempty. We first note that as in 2.1 we have:

THEOREM 7.1. *If a family $\mathscr{F} \in A(N)$ is $n+1$-connected, then it is strongly n-connected.*

PROOF. Let $A \in [N]^n$ and let $\mathscr{G} = \{F \in \mathscr{F} : A \subset F\}$. For each $m \in N$ there must exist, by $n+1$-connectivity, a set F belonging to $\mathscr{F}$, and therefore to $\mathscr{G}$ such that $A \cup \{m\} \subset F$. Hence $\cup \mathscr{G} = N$ so $\mathscr{G}$ cannot be finite. □

As in 2.2 we shall define a family $\mathscr{F}$ to be *weakly n-connected* if it is n-connected, but not strongly n-connected; *sharply n-connected* if it is strongly n-connected but not $n+1$-connected; and *fully connected* if it is strongly n-connected for every $n \in N$.

It is clear that because the notion of connectedness deals only with finite sets, it is in no way affected by *separability, i.e. we may connect or disconnect a family at will without affecting its *separability properties. However, to connect or disconnect families while preserving their separability properties is more difficult.

THEOREM 7.2. *If there exists a family* $\mathscr{F} = \{F_\alpha : \alpha \in \kappa\} \in M(N)$ *which is weakly n-separable, strongly n-separable, fully separable, fully Ramsey, or completely separable, then:*

a) *There exists a family* $\mathscr{G} \in M(N)$ *having the same separation properties as* $\mathscr{F}$ *and such that for each finite set* $A \subset N$ *the family* $\{G \in \mathscr{G} : A \subset G)\}$ *has cardinality* κ.

b) *For each* $m \in N$ *there exists a set* $A \in [N]^{m+1}$ *and families* $\mathscr{G}_m, \mathscr{H}_m \in M(N)$ *which have the same separation properties as* $\mathscr{F}$ *and satisfy:*

1) *The family* $\{G \in \mathscr{G}_m : A \subset G\}$ *has exactly one element while the family* $\{H \in \mathscr{H}_m : A \subset H\}$ *is empty and*

2) *For each set* $B \in [N]^{m+1}$ *other than A both of the families* $\{G \in \mathscr{G}_m : B \subset G\}$ *and* $\{H \in \mathscr{H}_m : B \subset H\}$ *have cardinality* κ.

Thus $\mathscr{G}$ *is fully connected while for each* $m \in N$ *the family* $\mathscr{G}_m$ *is weakly* $m+1$-*connected and the family* $\mathscr{H}_m$ *is sharply m-connected.*

PROOF. Although the proofs are similar in all cases, those involving weak or sharp separability are slightly more complicated, so we shall treat one of these in full detail and merely mention the appropriate modifications which would be necessary in the other cases. Thus suppose $\mathscr{F} = \{F_\alpha : \alpha \in \kappa\}$ is weakly n-separable. Choose any $\mathscr{D} \in \boldsymbol{D}_n(N)$ such that $\mathscr{F} / \mathscr{D}$ is finite. (If $\mathscr{F}$ were sharply n-separable, we would choose $\mathscr{D} \in \boldsymbol{D}_{n+1}(N)$ such that $\mathscr{F} / \mathscr{D} = \varnothing$; in the other cases no such decomposition would be required.) Now for each $\alpha \in \kappa$ decompose F_α into two infinite almost-disjoint sets R_α and S_α such that if $\{F_\alpha\} / \mathscr{D} = \varnothing$, then $\{R_\alpha, S_\alpha\} / \mathscr{D} = \varnothing$. Let $\mathscr{R} = \{R_\alpha : \alpha \in \kappa\}$ and $\mathscr{S} = \{S_\alpha : \alpha \in \kappa\}$. We note that $\mathscr{R} \cup \mathscr{S}$ is a maximal amily having the same separability properties as $\mathscr{F}$, and that we can think of these properties as being carried by either one of these families. Thus let f be any complete function from κ onto the set of all finite subsets of N. Then the family $\mathscr{G} = \{ R_\alpha \cup f(\alpha) : \alpha \in \kappa \} \cup \mathscr{S}$ satisfies part a. of the theorem.

Now choose any set $A \in [N]^{m+1}$ which is entirely contained in one member of $\mathscr{D}$ and let $\mathscr{A} = \{G \in \mathscr{G} : A \subset G\}$. By our construction $\mathscr{A}$ has cardinality κ. Let $\mathscr{B} = \{B \in [N]^{m+1} : B \neq A \wedge \mathrm{cd}\{G \in \mathscr{A} : B \subset G\} = \kappa\}$. Because $\mathscr{B}$ is countable, we may choose a complete function f from $\mathscr{A}$ onto $\mathscr{B}$ such that for each $G \in \mathscr{A}$ we have $f(G) \subset G$. We also choose a function g from $\mathscr{A}$ onto A such that for each $G \in \mathscr{A}$ we have $g(G) \in A - f(G)$. It now follows easily that we may set $\mathscr{H}_m = \{G - \{g(G)\} : G \in \mathscr{A}\} \cup (\mathscr{G} - \mathscr{A})$. Furthermore, if we choose some fixed element $G_0 \in \mathscr{A}$, then we may set $\mathscr{G}_m = (\mathscr{H}_m - \{G_0 - \{g(G_0)\}\}) \cup \{G_0\}$. □

8. Relative existence proofs

We are now ready to construct families having the various properties we have discussed. Our basic procedure will be first to well order an appropriate collection of families or sets such as $\boldsymbol{D}_n(N)$ or the power set of N, etc., and then to construct the required family, set by set, each set corresponding in some way to the appropriate member of the collection. In order to guarantee that the process does not terminate too soon due to the creation of a maximal family we shall begin our constructions with the selection of a countable family $\mathscr{F} \in A(N)$ and then shall assume that *no* member of $\boldsymbol{M}(N)$ has cardinality less than $2^{\aleph_0}$. As we have stated earlier this last assumption follows easily from either the continuum hypothesis or Martin's axiom and is therefore known to be relatively consistent with the axioms of Zermelo-Fraenkel set theory. This assumption is, of course, equivalent to the assumption that for every countably infinite set S, every member of $\boldsymbol{M}(S)$ has cardinality $2^{\aleph_0}$.

THEOREM 8.1. *If every member of $\boldsymbol{M}(N)$ has cardinality $2^{\aleph_0}$, then for each $n > 1$ there exist both weakly n-separable and sharply n-separable maximal almost-disjoint families.*

PROOF. By 2.3 and 3.3, it is sufficient to prove that there exist weakly n-separable families for each $n > 1$.

Choose any $n > 1$, any $\mathscr{D} = \{D_1, \cdots, D_n\} \in \boldsymbol{D}_n(N)$ such that each $D \in \mathscr{D}$ is infinite, any partition $\mathscr{S} = \{S_i : i \in \omega\}$ of N such that for each $S \in \mathscr{S}$ and each $D \in \mathscr{D}$ the set $S \cap D$ is infinite, and any function f from $2^{\aleph_0} - \omega$ onto $\boldsymbol{D}_n(N) - \{\mathscr{D}\}$. From these we construct a sequence $\{F_\alpha \subset N : \alpha \in 2^{\aleph_0}\}$ by transfinite induction. First, let $F_0 = S_0 \cap D_1$ and for each $n \in \omega - \{0\}$ let $F_n = S_n$. Now, assuming that we have constructed F_β for all $\beta < \alpha$ in such a way as to insure that, among other things, the family $\mathscr{F}_\alpha = \{F_\beta : \beta \in \alpha\}$ is almost disjoint, we construct F_α. Suppose $f(\alpha) = \mathscr{E} = \{E_1, \cdots, E_n\}$. Because $f(\alpha) \neq \mathscr{D}$ there must exist at least one set $E \in \mathscr{E}$ which intersects two different members of $\mathscr{D}$. Thus neither the family $\mathscr{E}^+ = \mathscr{E} - \mathscr{E}/\mathscr{D}$ nor the family $\mathscr{D}^+ = \mathscr{D} - \{D \in \mathscr{D} : \exists E \in \mathscr{D} E \subseteq D\}$ is empty. Further, it follows from our definition that $\cup \mathscr{D}^+ \subseteq \cup \mathscr{E}^+$. Because of the construction of $\{F_i : i \in \omega\}$, the family $\mathscr{F}_\alpha{}^+ = \{F_\beta \cap \cup \mathscr{D}^+ : \beta \in \alpha \text{ and } F_\beta \cap \cup \mathscr{D}^+ \neq^* \varnothing\}$ is infinite as well as almost disjoint and therefore belongs to $A(\cup \mathscr{D}^+)$. It follows easily from the hypothesis of the theorem that $\mathscr{F}_\alpha^+$ is not maximal over $\cup \mathscr{D}^+$, so we can choose an infinite set $G \subseteq \cup \mathscr{D}^+$ which is almost disjoint from every member of $\mathscr{F}_\alpha^+$. But $G \subseteq \cup \mathscr{D}^+ \subseteq \cup \mathscr{E}^+$, G is infinite, and $\mathscr{E}^+$ is finite, so there must exist at least one $E \in \mathscr{E}^+$ such that $G \cap E$ is infinite. Since $E \in \mathscr{E}^+$, E cannot be contained in any one member of $\mathscr{D}$. Thus there exist points $m, n \in E$ which are in different members of $\mathscr{D}$. Let $F_\alpha = (G \cap E) \cup \{m, n\}$. Then F_α is a subset of $E \in \mathscr{E}$ but is not a subset of any set $D \in \mathscr{D}$. Now let $\mathscr{F} = \{F_\alpha : \alpha \in 2^{\aleph_0}\}$. $\mathscr{F}$ is clearly n-separable and $\mathscr{F}/\mathscr{D} = \{F_0\}$, so $\mathscr{F}$ is weakly n-separable. □

We note that the above construction can be modified in an interesting way by requiring that f be complete.

Let $\mathscr{G}$ be the family $\mathscr{F} - \{F_0\}$ and let $\mathscr{F}^*$ and $\mathscr{G}^*$ be the maximal families constructed in 2.3 which preserve the weak n-separability of $\mathscr{F}$ and the sharp $n-1$-separability of $\mathscr{G}$. We see that $\mathscr{F}^*$ has the property that while $\mathscr{F}^*/\mathscr{D}$ has exactly one element, for every other decomposition $\mathscr{E} \in \boldsymbol{D}_n(N)$ the family $\mathscr{F}^*/\mathscr{E}$ has cardinality $2^{\aleph_0}$. Similarly, $\mathscr{G}^*/\mathscr{D}$ is empty, but again for each decomposition $\mathscr{E} \in \boldsymbol{D}_n(N) - (\mathscr{D}\}$ the family $\mathscr{G}^*/\mathscr{E}$ has cardinality $2^{\aleph_0}$.

Completing Section 5 we have:

THEOREM 8.2. *If every member of* $\boldsymbol{M}(N)$ *has cardinality* $2^{\aleph_0}$, *then there exists a fully Ramsey, completely separable family* $\mathscr{F} \in \boldsymbol{M}(N)$.

PROOF. Let $\mathscr{F}_\omega = \{F_i : i \in \omega\}$ be any partition of N into ω disjoint infinite sets, let f be any function from $2^{\aleph_0} - \omega$ onto the set of all infinite subsets of N, and let g be any function from $2^{\aleph_0}$ onto $\cup\{\boldsymbol{D}([N]^m) : m \in \omega\}$. We shall extend $\mathscr{F}_\omega$ to a family $\mathscr{F} = \{F_\alpha : \alpha \in 2^{\aleph_0}\}$ by transfinite induction. Assume that for some $\alpha \in 2^{\aleph_0} - \omega$ we have already defined $\mathscr{F}_\alpha = \{F_\beta \subset N : \beta \in \alpha\}$ in a way such that, among other things, we know that it is almost disjoint. If $\mathscr{F}_\alpha$ finitely dominates

$f(\alpha)$, then we set $F_\alpha = F_0$. Otherwise, it follows from the construction and the hypothesis of the theorem that the family $\mathscr{F}_\alpha^+ = \{F_\beta \cap f(\alpha)\colon \beta \in \alpha \wedge F_\beta \cap f(\alpha) \neq^* \varnothing\}$ belongs to $A(f(\alpha))$, but is not maximal. Thus we may choose a set $A_\alpha \subseteq f(\alpha)$ which is infinite but nevertheless almost disjoint from every member of $\mathscr{F}_\alpha^+$ and therefore from every member of $\mathscr{F}_\alpha$. Let β be the least ordinal such that for some $m \in M$ we have $g(\beta) \in D([N]^m)$ and $|\mathscr{F}_\alpha|^m / g(\beta) = \varnothing$. By Ramsey's theorem we may choose an infinite subset F_α of A such that $\exists D \in g(\beta)\ [F_\alpha]^m \subseteq D$. It now follows easily that $\mathscr{F} = \{F_\alpha\colon \alpha \in 2^{\aleph_0}\}$ is fully Ramsey and completely separable. □

Again we can strengthen our theorem somewhat by insisting that f and or g be complete, but the results are not as striking as in the case of 8.1.

THEOREM 8.3. *If every member of $M(N)$ has cardinality $2^{\aleph_0}$, then for every natural number n there exists a sharply n-*separable family $\mathscr{F} \in M(N)$.*

PROOF. By 6.2 it is sufficient to show that for every $n > 1$ we can find a decomposition $\mathscr{D} \in D_{n+1}(N)$ and an n-*separable family $\mathscr{F} \in A(N)$ which does not *separate over $\mathscr{D}$. As before, we choose a partition $\mathscr{F}_\omega = \{F_i\colon i \in \omega\}$ of N into disjoint infinite sets, a decomposition $\mathscr{D} = \{D_0, \cdots, D_n\} \in D_{n+1}(N)$ such that for each $D \in \mathscr{D}$ and each $F \in \mathscr{F}_\omega$ we have $D \cap F \neq^* \varnothing$, and a function f from $2^{\aleph} - \omega$ onto $D_n(N)$. Again, as before, we extend $\mathscr{F}_\omega$, by induction, to a family $\mathscr{F} = \{F_\alpha\colon \alpha \in 2^{\aleph_0}\}$. Assume we have constructed F_β for all $\beta \in \alpha$ in such a manner that $\mathscr{F}_\alpha = \{F_\beta\colon \beta \in \alpha\}$ is almost disjoint, $\mathscr{F}_\alpha /^* \mathscr{D} = \varnothing$, and for each $\beta \in \alpha - \omega$ we have $F_\beta \in \mathscr{F}_\alpha / f(\beta)$. From the construction of $\mathscr{F}_\omega$ it follows that for each $i \leqq n$ the family $\mathscr{F}_\alpha^i = \{F_\beta \cap D_i\colon \beta \in \alpha \wedge F_\beta \cap D_i \neq^* \varnothing\}$ is infinite and therefore by the induction hypothesis belongs to $A(D_i)$. Since, by the hypothesis of the theorem, each $\mathscr{F}_\alpha^i$ is too small to be maximal, there must exist infinite sets $G_0, G_1, \cdots, G_n$ such that for each $i \leqq n$ we have $G_i \subseteq D_i$ and G_i is almost disjoint from every member of $\mathscr{F}_\alpha^i$ and therefore every member of $\mathscr{F}_\alpha$. But since $f(\alpha)$ contains fewer than $n+1$ members, there must exist at least one $E \in f(\alpha)$ such that for some $i \neq j$ we have $E \cap G_i$ and $E \cap G_j$ both infinite. Thus we may set $F_\alpha = (G_i \cup G_j) \cap E$.

□

Finally, we show that, at least under our extra hypothesis, neither 4.1 nor even 4.2.a can be extended to weakly 2-separable families.

THEOREM 8.4. *If every member of $M(N)$ has cardinality $2^{\aleph_0}$, then there* *exists a weakly 2-separable family* $\mathscr{F} \in \boldsymbol{M}(N)$ *and a set* $F \in \mathscr{F}$ *such that* $G \in \mathscr{F}$ *implies that* $G \cap F \neq \varnothing$.

PROOF. Let $\mathscr{F}_\omega = \{F_i : i \in \omega\}$ be a partition of N into infinite almost-disjoint sets such that $i \in \omega \rightarrow F_i \cap F_0 \neq \varnothing$ and let f be any function from $2^{\aleph_0} - \omega$ onto $\boldsymbol{D}_2(N)$. We now extend $\mathscr{F}_\omega$ to a family $\mathscr{F}^+ = \{F_\alpha : \alpha \in 2^{\aleph_0}\} \in A(N)$ by induction. Assume for some $\alpha \geqq \omega$ we have already defined $\mathscr{F}_\alpha = \{F_\beta : \beta \in \alpha\}$ such that $\mathscr{F}_\alpha \in A(N)$ and suppose $f(\alpha) = \mathscr{D} = \{D_1, D_2\}$. If $F_0 \subseteq D_1$ or $F_0 \subseteq D_2$ define $F_\alpha = F_0$. Otherwise, since by hypothesis of the theorem $\mathscr{F}_\alpha$ cannot be maximal, there must exist an infinite set $F \subset N$ which is almost disjoint from every member of $\mathscr{F}_\alpha$. But then there must exist a $D \in \mathscr{D}$ such that $F \cap D$ is infinite. Now choose any point $n \in F_0 \cap D$ and let $F_\alpha = \{n\} \cup (F \cap D)$. While it is clear that the family $\mathscr{F}^+ = \{F_\alpha : \alpha \in 2^{\aleph_0}\}$ belongs to $A(N)$, is 2-separable, and has the property that $F \in \mathscr{F}^+ \rightarrow F \cap F_0 \neq \varnothing$, it does not follow that it is necessarily maximal. However, if not, choose any family $\mathscr{G}$ such that $\mathscr{F}^+ \cup \mathscr{G} \in \boldsymbol{M}(N)$ and any element $n \in F_0$ and let $\mathscr{F} = \mathscr{F}^+ \cup \{G \cup \{n\} : G \in \mathscr{G}\}$. □

9. Martin's axiom

So far we have, in our proofs, required only the standard axioms of $ZF + AC$ plus at most a hypothesis that every member of $A(N)$ of cardinality less than $2^{\aleph_0}$ is not maximal. For our remaining results this does not appear to be strong enough. The continuum hypothesis, on the other hand, turns out to be much too strong. What we shall use is an axiom due to D. A. Martin [5] which is known to be relatively consistent with $ZF + AC$; i.e. if $ZF + AC$ is itself consistent so is $ZF + AC +$ Martin's axiom [7]. Since the continuum hypothesis implies Martin's axiom but not conversely, our results will be strictly stronger than if we had used the continuum hypothesis.

For the convenience of the reader we shall state Martin's axiom here.

Suppose $\langle A, <_A \rangle$ is a partial order structure. If for any uncountable set $B \subseteq A$ there exist two distinct elements $b, c \in B$ which admit a common upper bound d in A (i.e. $b <_A d$ and $c <_A d$) we shall say $\langle A, <_A \rangle$ has the *countable antichain condition* (c.a.c.). We define a set $B \subseteq A$ to be *open* if $b \in B \wedge b <_A c \rightarrow c \in B$ and to be *dense* if $a \in A \rightarrow \exists b \in B \; a <_A b$. Finally, if $\mathscr{F}$ is any family of subsets of A, then we shall call a set $G \subseteq A$ an *$\mathscr{F}$-generic filter* if G intersects every member of $\mathscr{F}$, for any $f, g \in G$ there is an $h \in G$ such that $f <_A h$ and $g <_A h$, and $a <_A f \in G$ implies $a \in G$. Using the above, Martin's axiom is:

AXIOM 9.1. (MARTIN). *If $\langle A, <_A \rangle$ is any partial order structure satisfying the countable antichain condition and $\mathscr{F}$ is any family of fewer than $2^{\aleph_0}$ open dense subsets of A, then there exists an $\mathscr{F}$-generic filter in A.*

In our applications we shall confine our use of Martin's axiom to lemmas which the reader may prove directly from the continuum hypothesis using standard diagonalization techniques. Our first lemma is slightly more general than will be necessary because it is of interest in its own right.

LEMMA 9.2. *Suppose Martin's axiom holds. Then for any $n \in N$ and any infinite $\mathscr{F} \in A(N)$ of cardinality less than $2^{\aleph_0}$ there exists an infinite set $G \subset N$ which is almost disjoint from every member of $\mathscr{F}$ but which intersects each member of $\mathscr{F}$ in at least n points.*

PROOF. Suppose $\mathscr{F}$ and n are given. To apply Martin's axiom we must first construct a partial order structure $\langle A, <_A \rangle$. Let $B = \{\langle f, S \rangle : f$ is a finite subset of N and S is a finite subset of $\mathscr{F} \times N\}$. Thinking of f as a subset of the set which will be extended to become G and $\langle F, m \rangle \in S$ as a bound on the eventual intersection of F and G, we define a pair $\langle f, S \rangle \in B$ to be consistent iff for every pair $\langle F, m \rangle \in S$ the set $f \cap F$ has fewer than $m + n$ elements. Now let $A = \{b \in B \colon b$ is consistent$\}$ and let $\langle f, S \rangle <_A \langle g, T \rangle$ hold iff $f \subseteq g$ and $S \subseteq T$.

We first note that if both $\langle f, S \rangle$ and $\langle f, T \rangle$ belong to A, so does $\langle f, S \cup T \rangle$. Since this is a common upper bound and since the set of finite subsets of N is countable, we see that $\langle A, <_A \rangle$ satisfies the countable antichain condition.

For each $m \in N$ let $H_m = \{\langle f, S \rangle : f$ has at least m elements$\}$. Each H_m is clearly open and, because $\mathscr{F}$ is infinite, dense. Also, for each $F \in \mathscr{F}$ let $H_F = \{\langle f, S \rangle : f \cap F$ has at least n elements and for some $m \in N$ the pair $\langle F, m \rangle$ belongs to $S\}$. Again, for the same reasons, each H_F is open and dense. Hence

$$\mathscr{H} = \{H_m \colon m \in N\} \cup \{H_F \colon F \in \mathscr{F}\}$$

is a family of fewer than $2^{\aleph_0}$ open dense subsets of A. Thus by Martin's axiom there exists an $\mathscr{H}$-generic filter $H \subset A$. It is now easy to see that the set

$$G = \cup \{f : \exists S \langle f, S \rangle \in H\}$$

satisfies the conditions of the lemma. □

COROLLARY 9.3. *Martin's axiom implies that every member of $M(N)$ has cardinality $2^{\aleph_0}$.*

THEOREM 9.4. *If Martin's axiom holds, then every 2-separable family $\mathscr{F} \in A(N)$ has cardinality $2^{\aleph_0}$.*

PROOF. Let $\mathscr{F} \in A(N)$ have cardinality less than $2^{\aleph_0}$. Then by 9.2 there exists an infinite set $G \subset N$ such that $\mathscr{F} \cup \{G\} \in A(N)$ and for every $F \in \mathscr{F}$, $F \cap G$ has at least two elements. But this, by 4.3a, implies that $\mathscr{F} \cup \{G\}$, and therefore $\mathscr{F}$ itself, cannot be 2-separable. □

Finally we mention one interesting non-separable family. While it is easy to construct trivial non-separable families it would be of interest to find classes of "very" non-separable families. The following example was motivated by [1,2]. Let Q be the set of all rational numbers. We shall refer to a family $\mathscr{F} \in \boldsymbol{P}(Q)$ as *everywhere dense* iff each member $F \in \mathscr{F}$ is everywhere dense with respect to the standard topology of the real line and we shall prove that there exists an everywhere-dense family $\mathscr{F} \in M(Q)$. Again we first use Martin's axiom to prove a lemma.

LEMMA 9.5. *If Martin's axiom holds and if $\mathscr{F} \in A(Q)$ is an infinite everywhere-dense family of cardinality less than $2^{\aleph_0}$, then there exists an infinite everywhere-dense set $G \subset Q$ which is almost disjoint from every member of $\mathscr{F}$.*

PROOF. As in the proof of 9.2 we first construct a partial order structure $\langle A, <_A \rangle$. Let $B = \{\langle f, S \rangle : f$ is a finite subset of Q and S is a finite subset of $\mathscr{F} \times N\}$. Again we will think of a set f appearing in a member of B as a subset of the set G we are in the process of constructing, and a pair $\langle F, n \rangle$ as an upper bound for the number of elements in the set $F \cap G$, so we define an element $\langle f, S \rangle$ to be consistent iff for each $\langle F, n \rangle \in S$ the set $F \cap f$ has at most n elements. Now again we define A to be the set $\{b \in B : b$ is consistent$\}$ and for $\langle f, S \rangle$ and $\langle g, T \rangle$ in A we again define $\langle f, S \rangle <_A \langle g, T \rangle$ to hold iff $f \subseteq g$ and $S \subseteq T$.

As in our earlier construction, if $\langle f, S \rangle$ and $\langle f, T \rangle$ belong to A so does $\langle f, S \cup T \rangle$; thus $\langle A, <_A \rangle$ satisfies the countable antichain condition. Now, for each $F \in \mathscr{F}$ we define $R_F = \{\langle f, S \rangle \in A : \exists n \langle F, n \rangle \in S\}$, for each $n \in N$ we define $R_n = \{\langle f, S \rangle : f$ has at least n elements$\}$, and for each $p, q \in Q$ such that $p < q$ we define $R_{pq} = \{\langle f, S \rangle : \exists r \in f \; p < r < q\}$. Finally, let

$$\mathscr{G} = \{R_F : F \in \mathscr{F}\} \cup \{R_n : n \in N\} \cup \{R_{pq} : p, q \in Q \text{ and } p < q\}.$$

We note that since each member of $\mathscr{G}$ is open and dense, and $\mathscr{G}$ has cardinality less than $2^{\aleph_0}$, Martin's axiom allows us to choose a $\mathscr{G}$-generic filter $H \subset A$. It then follows that the set $G = \cup \{g : \exists S \langle g, S \rangle \in H\}$ satisfies the conditions of the lemma. □

We can now prove:

THEOREM 9.6. *Martin's axiom implies the existence of an everywhere-dense family* $\mathscr{F} \in M(Q)$.

PROOF. Let $\{F_i: i\in \omega\}$ be an infinite family of everywhere-dense pairwise disjoint subsets of Q. Using the lemma and transfinite induction we can extend this to an everywhere-dense family $\mathscr{G} = \{G_\alpha: \alpha\in 2^{\aleph_0}\} \in A(Q)$. If G is maximal we are done; otherwise, choose any family $\mathscr{H}$ such that $(\mathscr{G} \cup \mathscr{H}) \in M(Q)$. We may assume that $\mathscr{H}$ has cardinality $2^{\aleph_0}$ and can therefore be expressed as $\{H_\alpha: \alpha \in 2^{\aleph_0}\}$. Now let $\mathscr{F} = \{G_\alpha \cup H_\alpha: \alpha\in 2^{\aleph_0}\}$. □

10. Topological applications

Let N be the topological space consisting of N and the discrete topology and let βN be the Stone-Čech compactification of N. It is well known [3] that $\beta N - N$ can be represented by the set of all non-principal ultrafilters over N with the topology generated by the following basis $\mathfrak{A}$. For each $A \subseteq N$ let $A^\beta = \{u \in \beta N - N: A \in u\}$ and let $\mathfrak{A} = \{A^\beta: A \subseteq N\}$. We see immediately:

THEOREM 10.1. *For any* A, $B \subseteq N$:

a) $A^\beta \cup B^\beta = (A \cup B)^\beta$

b) $A^\beta \cap B^\beta = (A \cap B)^\beta$

c) $(\beta N - N) - A^\beta = (N - A)^\beta$

d) $A^\beta \subset B^\beta \leftrightarrow A \subset^* B$

e) $A^\beta = B^\beta \leftrightarrow A =^* B$

PROOF. Left to reader. □

It is also known that a set $S \subseteq \beta N - N$ is clopen iff it belongs to $\mathfrak{A}$. Thus we may look upon an almost-disjoint family $\mathscr{F}$ as "representing" a family $\mathscr{F}^\beta = \{F^\beta: F \in \mathscr{F}\}$ of disjoint subsets of $\beta N - N$. It then follows immediately that $\mathscr{F}$ is maximal iff $\cup \mathscr{F}^\beta$ is everywhere dense. (This application of almost-disjoint families was suggested to the author by W. W. Comfort in a private communication.) While we see no direct applications of separability much less Ramsey properties, we can extend *separability properties obtaining:

THEOREM 10.2. *There exists a family $\mathscr{F}$ of disjoint clopen subsets of $\beta N - N$ such that $\cup \mathscr{F}$ is everywhere dense, and for every decomposition $\mathscr{D}$* of *$\beta N - N$ into finitely many clopen sets the family $\mathscr{F} | \mathscr{D}$ is infinite.*

PROOF. This is simply a restatement of 3.2. □

From Section 8 we have:

THEOREM 10.3. *If every infinite maximal family of disjoint clopen subsets of* $\beta N - N$ *has cardinality* $2^{\aleph_0}$, *then for each* $n \in N$ *there exists an infinite maximal family* $\mathscr{F}$ *of disjoint clopen subsets of* $\beta N - N$ *satisfying:*

a) *For every decomposition of* $\mathscr{D}$ *of* $\beta N - N$ *into* n *clopen sets the family* $\mathscr{F}/\mathscr{D}$ *is infinite, and*

b) *There exists a decomposition* $\mathscr{E}$ *of* $\beta N - N$ *into* $n+1$ *clopen sets such that the family* $\mathscr{F}/\mathscr{E}$ *is empty.*

Proof. This is merely a direct interpretation of 8.3. □

Finally, we can also interpret and, in fact, extend 8.2 to:

THEOREM 10.4. *If every infinite maximal family of disjoint clopen subsets of* $\beta N - N$ *has cardinality* $2^{\aleph_0}$, *then there exists a family* $\mathscr{F}^\beta$ *of disjoint clopen subsets of* $\beta N - N$ *such that for each* $u \in (\beta N - N) - \cup \mathscr{F}^\beta$ *and each open subset* O *of* $\beta N - N$ *containing* u *the family* $\mathscr{F}^\beta/O$ *has cardinality* $2^{\aleph_0}$.

PROOF. Let $\mathscr{F}$ be the completely separable family constructed in 8.2, and look at any $u \in \beta N - N$ and any open set O containing u. Because $\mathfrak{A}$ is a basis, there must exist a set $A \subseteq N$ such that $u \in A^\beta \subseteq O$. Suppose $\mathscr{F}$ finitely dominates A. Then there exists a finite family $\mathscr{G} \subset \mathscr{F}$ such that $A \subseteq \cup \mathscr{G}$. But, by 10.1, this implies that $u \in A^\beta \subseteq (\cup \mathscr{G})^\beta = \cup \mathscr{G}^\beta \subseteq \cup \mathscr{F}^\beta$. But if $\mathscr{F}$ does not infitely dominate A, then by the remark following 8.2, we may assume that $\mathscr{F}/A$ has cardinality $2^{\aleph_0}$. Thus $\mathscr{F}^\beta/O \supseteq \mathscr{F}^\beta/A^\beta$ must also have cardinality $2^{\aleph_0}$. □

REFERENCES

1. G. Choquet, *Construction d'ultrafiltres sur N*, Bull. Sci. Math. **92,** 2nd ser. (1968), 41–48.

2. G. Choquet, *Deux classes remarquables d'ultrafiltres sur N*, Bull. Sci. Math. **92,** 2nd ser. (1968), 143–153.

3. L. Gillman and M. Jerison, *Rings of Continuous Functions*, D. Van Nostrand Company, Inc., Princeton, New Jersey (1960).

4. S. H. Hechler, *Short complete nested sequences in* βN − *N and small almost-disjoint families*, to appear in General Topology and its Applications.

5. D. A. Martin and R. M. Solovay, *Internal Cohen extensions*, Ann. of Math. Logic, **2** (1970), 143–178.

6. F. P. Ramsey, *On a problem of formal logic*, Proc. London Math. Soc. **30,** 2nd. ser. (1929), 264–286.

7. R. M. Solovay and S. Tennenbaum, *Iterated Cohen extensions and Souslin's problem*, to appear.

CASE WESTERN RESERVE UNIVERSITY